A MIP-Based Dual Bounding Technique for the Irregular Nesting Problem

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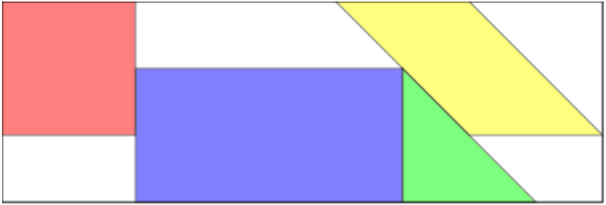
The Irregular Nesting Problem

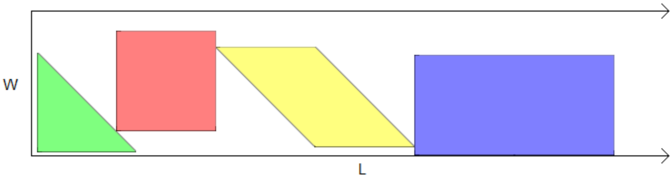
Given a set of 2D objects, arrange them such that:

- No two objects overlap.
- Required length is minimized.



Assumptions

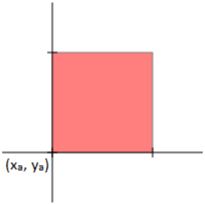
- All objects are convex polygons.
- We are "nesting" these objects into a rectangle of:
    - Fixed width (W).
    - Potentially infinite length (L).
- We do not allow rotation.



Fischetti-Luzzi Model: Decision Variables

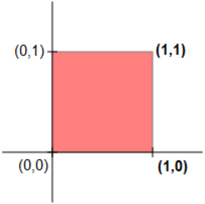
The model used in Fischetti and Luzzi (2009) and refined in Alvarez-Valdes, Martinez, and Tamarit (2013) has two components. The first is a Linear Program (LP) that minimizes the length of the outer region.

Our decision variables are the length of the outer region, L, and an offset for each polygon. For instance, polygon A, shown in red, is offset by the linear decision variables (x_a, y_a).



Fischetti-Luzzi Model: Parameters

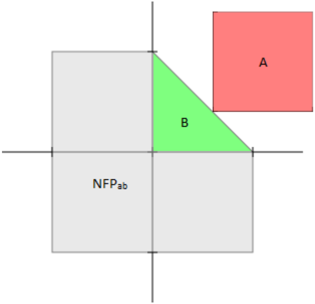
The model takes as input a fixed width, $W > 0$, and a clockwise set of vertex offsets for each polygon. Polygon A, shown in red, has the offsets $P_a = \{(0,0),(0,1),(1,1),(1,0)\}$.



We'll need the maximum x and y offsets for each polygon. In this case $x_a^{max} = y_a^{max} = 1$.

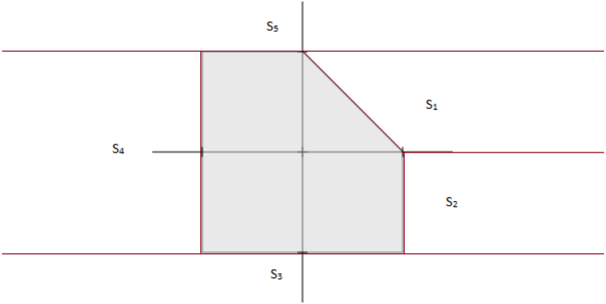
Fischetti-Luzzi Model: Overlap Elimination

The model eliminates overlap using the No-Fit Polygon (NFP). This is drawn by holding one object stationary and tracing another around it so that they touch but do not overlap. If $(x_a - x_b, y_a - y_b) \notin NFP_{ab}$, then A and B do not overlap.



Fischetti-Luzzi Model: Overlap Elimination

We enforce these offset relationships by slicing up the regions around the NFP as prescribed in Alvarez-Valdes, Martinez, and Tamarit (2013).



There are 5 regions outside NFP_{ab}. A binary variable associated with each region turns on and off its constraints.

Fischetti-Luzzi Model: Overlap Elimination

Consider region S_4 for NFP_{ab}. If $(x_a - x_b, y_a - y_b) \in S_4$, then the following constraints must be active.

$$x_a - x_b \leq -1$$

$$-1 \leq y_a - y_b \leq 1$$

If we have binary variables $b_1, \ldots, b_5$ such that $\sum_{i=1}^{5} b_i = 1$, then we can write the constraints for S_4 as:

$$x_a - x_b \leq -1 + M(1 - b_4)$$

$$y_a - y_b \geq -1 - M(1 - b_4)$$

$$y_a - y_b \leq 1 + M(1 - b_4)$$

Where M is an appropriately large value. We add constraint sets like this for each region and each pair of polygons.

Fischetti-Luzzi Model

We formulate the Fischetti-Luzzi model as follows.

$$
\begin{array}{lll}
\min & L & \\
\text{s.t.} & x_p + x_p^{max} \leq L & \forall\, p \\
& y_p + y_p^{max} \leq W & \forall\, p \\
& (x_p - x_q, y_p - y_q) \notin NFP_{pq} & \forall\, p < q \\
& x_p, y_p \geq 0 & \forall\, p
\end{array}
$$

The overlap elimination constraints are what makes this difficult. They add a binary variable and big-M for every edge for every pair of polygons!

Motivation

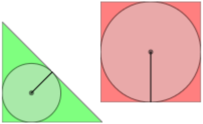
The Fischetti-Luzzi model grows in complexity too quickly to prove optimality for large numbers of objects.

Very effective heuristic techniques exist that can generate good solutions for large nesting problems, but these lack optimality bounds.

We would like a model that can generate dual bounds for the INP at low cost. In this talk we present preliminary results for ongoing research in that direction.

Approximating Polygons with Chebyshev Circles

One option is to replace the polygons with simpler objects. Fasano (2013) suggests approximating them with circles.



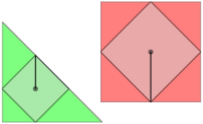
We replace polygons A and B with their maximum inscribed Chebyshev circles shown above. These have centers $\left(x_a + c_a^x, y_a + c_a^y\right)$ and $\left(x_b + c_b^x, y_b + c_b^y\right)$and radiuses r_a and r_b, respectively. We can now add an overlap elimination constraint:

$$\left\|\left(x_a + c_a^x, y_a + c_a^y\right) - \left(x_b + c_b^x, y_b + c_b^y\right)\right\|_2 \geq r_a + r_b$$

But this is non-convex, so we cannot use it for a dual bound.

Approximating Polygons with L1-Norm Balls

Alternatively, we can approximate polygons with L^1-norm balls.



We can eliminate overlap for these relaxations using mixed-integer linear constraints, where $d_x, d_y \geq 0$ and $b_x, b_y \in \{0, 1\}$.

$$d_x \leq (x_b + c_b^x) - (x_a + c_a^x) + Mb_x$$
$$d_y \leq (y_b + c_b^y) - (y_a + c_a^y) + Mb_y$$
$$d_x \leq (x_a + c_a^x) - (x_b + c_b^x) + M(1 - b_x)$$
$$d_y \leq (y_a + c_a^y) - (y_b + c_b^y) + M(1 - b_y)$$
$$d_x + d_y \geq r_1 + r_2$$

Dual Bounding Model

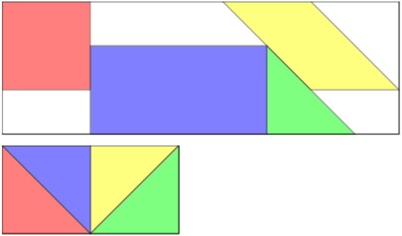
Thus our relaxed "dual bounding" model *(in the sense of bounding, not duality)*, where (c_p^x, c_p^y) is the center of the maximum inscribed L^1-norm ball of polygon p with respect to its offset (x_p, y_p), is:

$$
\begin{array}{lll}
\min & L & \\
\text{s.t.} & x_p + x_p^{max} \leq L & \forall\, p \\
& y_p + y_p^{max} \leq W & \forall\, p \\
& d_x^{pq} \leq \left(x_q + c_q^x\right) - \left(x_p + c_p^x\right) + Mb_x^{pq} & \forall\, p < q \\
& d_y^{pq} \leq \left(y_q + c_q^y\right) - \left(y_p + c_q^y\right) + Mb_y^{pq} & \forall\, p < q \\
& d_x^{pq} \leq \left(x_p + c_p^x\right) - \left(x_q + c_q^x\right) + M\left(1 - b_x^{pq}\right) & \forall\, p < q \\
& d_y^{pq} \leq \left(y_p + c_p^y\right) - \left(y_q + c_q^y\right) + M\left(1 - b_y^{pq}\right) & \forall\, p < q \\
& d_x^{pq} + d_y^{pq} \geq r_p + r_q & \forall\, p < q \\
& x_p, y_p \geq 0 & \forall\, p \\
& b_x^{pq}, b_y^{pq} \in \{0, 1\} & \forall\, p < q
\end{array}
$$

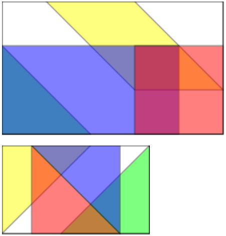
Dual Bounding Model: Trade-Off

Replacing polygons with L^1-norm balls allows a certain amount of overlap in the optimal solution. It also significantly reduces the number of binary variables. A couple examples:

Fischetti-Luzzi Model



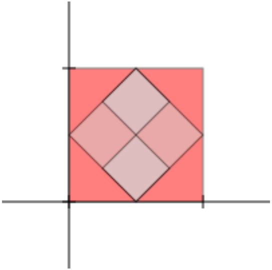
Dual Bounding Model

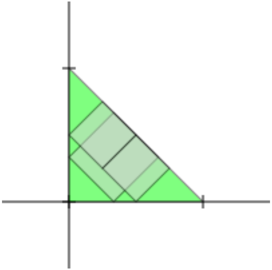


Dual Bounding Bisection Model

We can refine our dual bounding model by eliminating more overlap among the polygon pairs. One option:

- Bisect each polygon along the line intersecting its centroid and its closest point on the exterior of the polygon.
- Compute the L^1-norm ball for each sub-polygon.
- Add overlap elimination constraints as before, considering the sub-polygons paired with our original polygon centers.

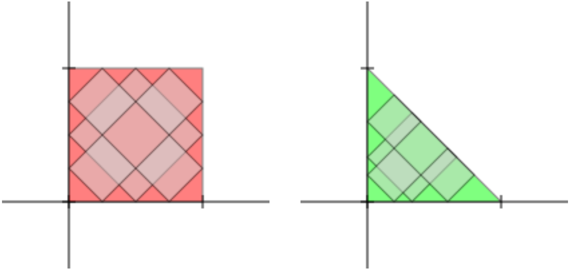




Dual Bounding Subdivision Model

Another option:

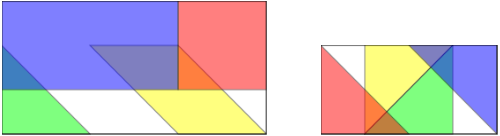
- Compute the centroid of each polygon.
- Subdivide each polygon by connecting the centroid to the midpoint of each edge.
- Compute the L^1-norm ball for each sub-polygon.
- Add overlap elimination constraints as before, considering the sub-polygons paired with our original polygon centers.



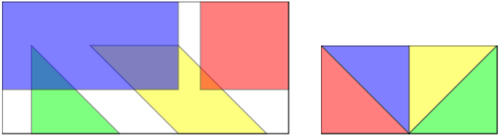
Bisection & Subdivision: Trade-Off

Bisection multiplies the number of binary variables in our model by 2. Subdivision multiplies them by 2 $\times$ the number of edges.

Bisection:



Subdivision:



Successive Approximation Models

We can start with either our "dual bounding bisection" or "dual bounding subdividision" models and improve them further using solver callbacks.

Formulate the model as before. When a new solution is found, do the following:

- For every pair of polygons, use their NFP to test for overlap.
- If they do, compute the overlapping region's L^1-norm ball.
- Add cuts to the model that eliminate this overlap.

Eliminating successively smaller amounts of overlap will eventually discover the optimal layout. But we pay a price. Each cut requires the addition of two binary variables.

Methodology

We run a series of test cases containing increasing numbers of convex polygons through each model. The solver is interrupted after an hour of computation, or when an optimal solution is found. We record bounds and incumbent solutions as they are discovered.

Software & hardware specifications:

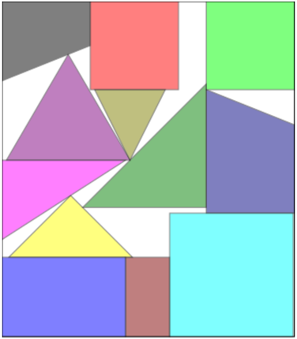
- MIP Solver: Gurobi v5.6
- Geometry: Generic Polygon Clipper (GPC) v2.32
- Machine: 4-core Intel i7 @ 2GHz, 8 GB RAM, Windows 7

Branching priority on binary variables corresponds to polygon size.

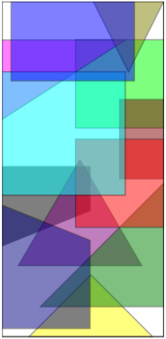
Test Instance: Fujita (12 misc objects)

This instance contains 12 convex polygons from Fujita (1993).

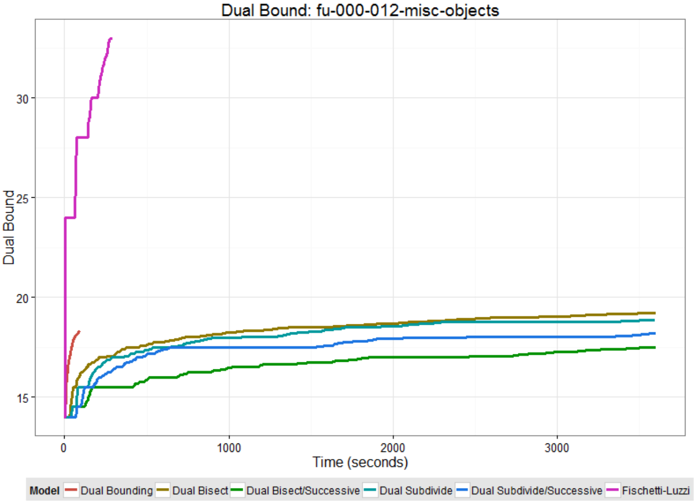
Fischetti-Luzzi Model Output



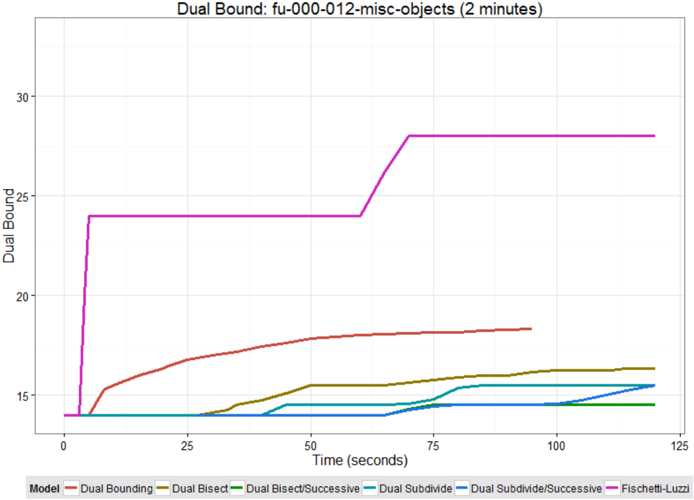
Dual Bounding Model Output



Test Instance: Fujita (12 misc objects)

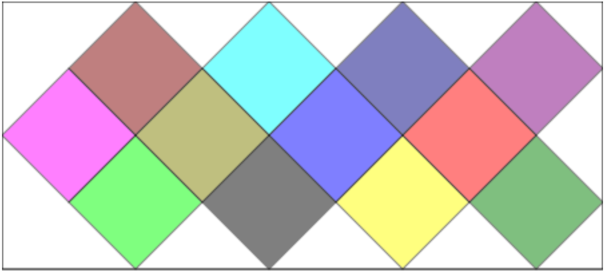


Test Instance: Fujita (12 misc objects)

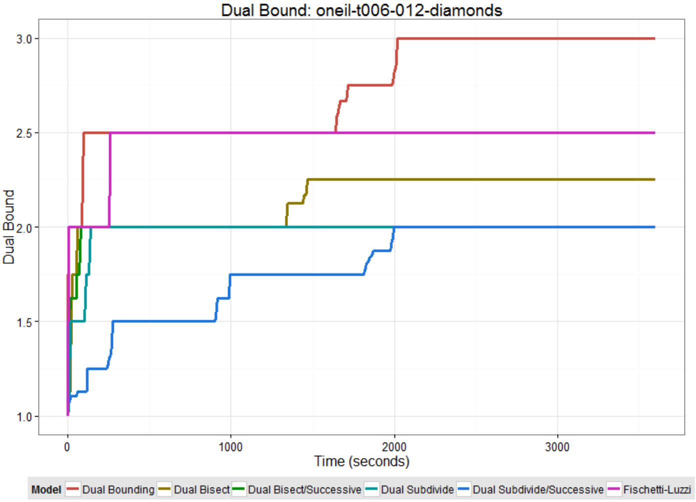


Test Instance: O'Neil (12 diamonds)

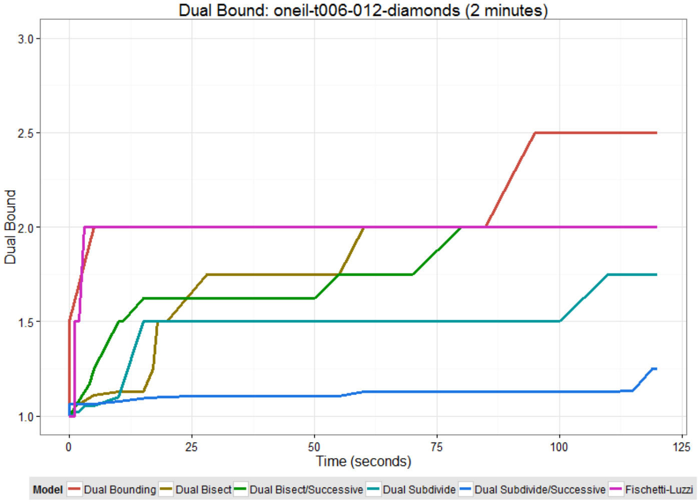
This instance contains 12 diamonds. In this case, the dual bounding model converges to the same solution as the Fischetti-Luzzi model.



Test Instance: O'Neil (12 diamonds)



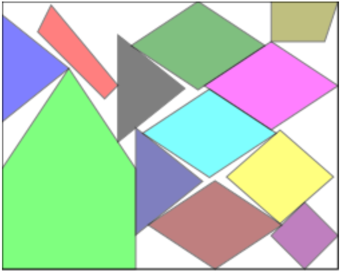
Test Instance: O'Neil (12 diamonds)



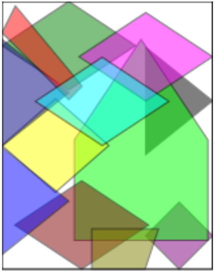
Test Instance: O'Neil (12 misc objects)

This instance contains 12 convex polygons.

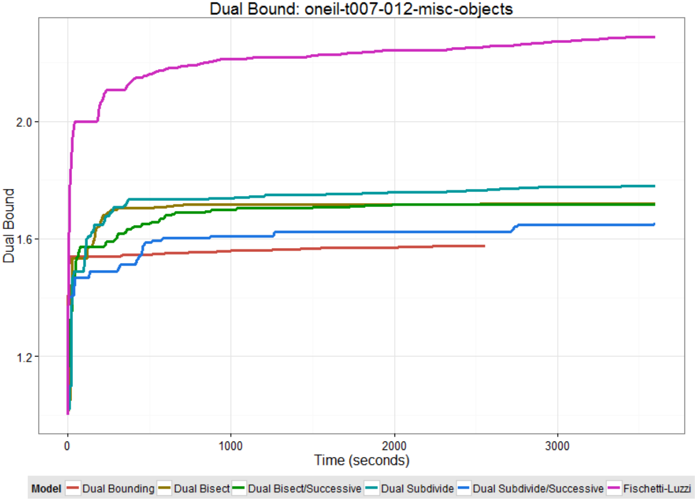
Fischetti-Luzzi Model Output



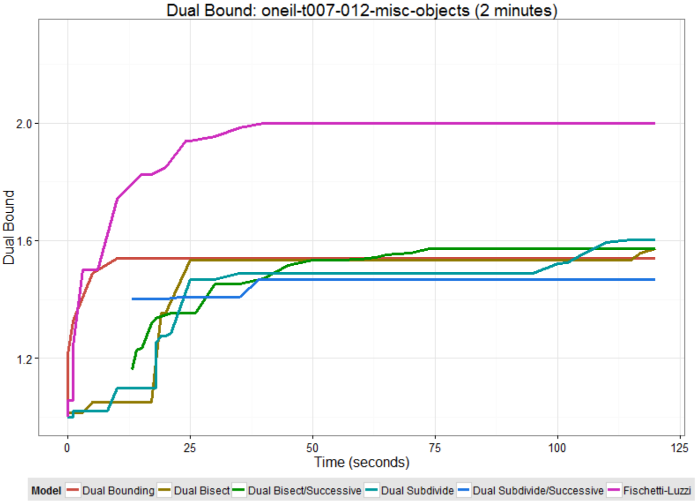
Dual Bounding Model Output



Test Instance: O'Neil (12 misc objects)



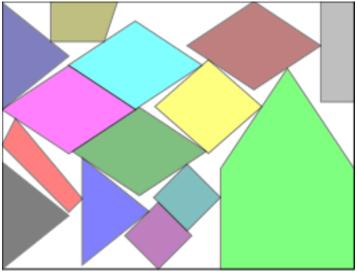
Test Instance: O'Neil (12 misc objects)



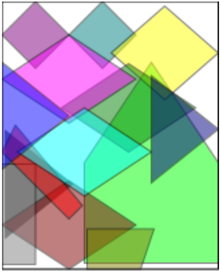
Test Instance: O'Neil (14 misc objects)

This instance contains 14 convex polygons.

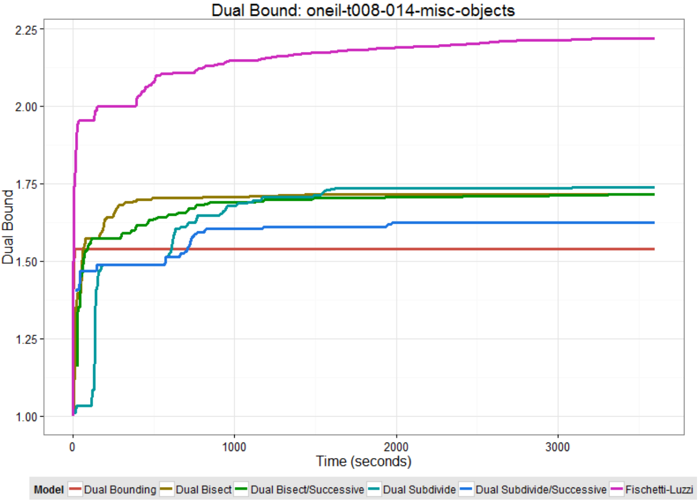
Fischetti-Luzzi Model Output



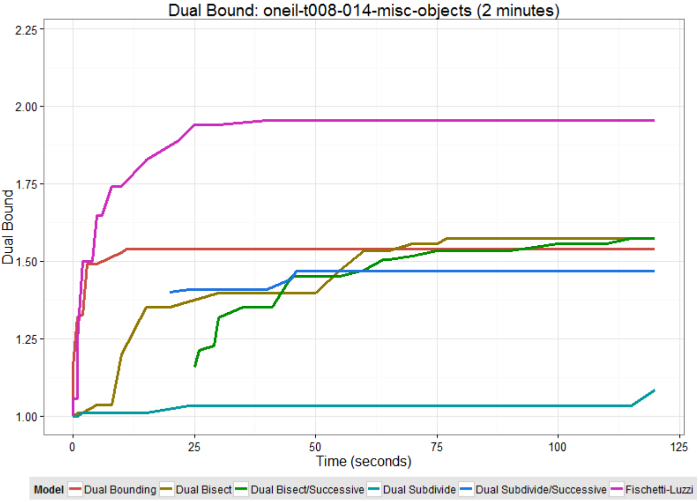
Dual Bounding Model Output



Test Instance: O'Neil (14 misc objects)



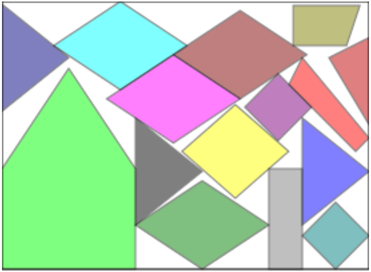
Test Instance: O'Neil (14 misc objects)



Test Instance: O'Neil (16 misc objects)

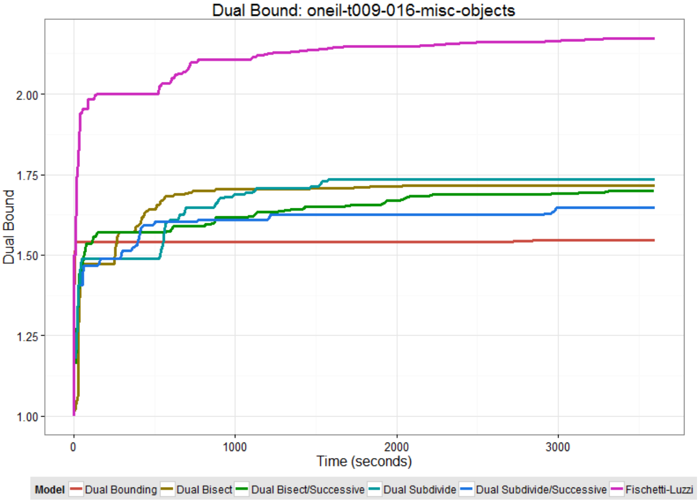
This instance contains 16 convex polygons.

Fischetti-Luzzi Model Output

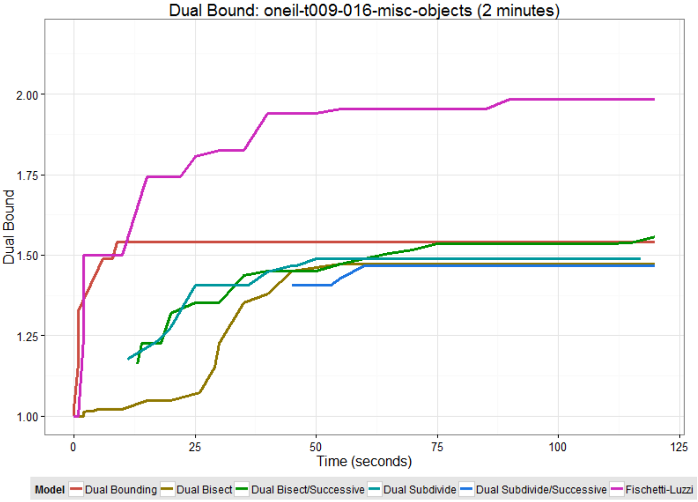


Dual Bounding Model Output

Test Instance: O'Neil (16 misc objects)



Test Instance: O'Neil (16 misc objects)



Time to Final Dual Bound

Time in seconds to converge to its final dual bound or optimal solution and the gap associated with that solution, per model and test instance. Solver is interrupted after an hour.

	Fischetti-Luzzi		Dual Bounding	
Test Instance	Time	Gap	Time	Gap
Fujita (12 misc)	295s	0%	95s	0%
O'Neil (12 diamonds)	260s	44.4%	2020s	33.3%
O'Neil (12 misc)	3600s	8.9%	2560s	0%
O'Neil (14 misc)	3330s	15.7%	451s	4.6%
O'Neil (16 misc)	3600s	20.9%	3065s	4.2%

Note: These models converge to different solutions.

Conclusions

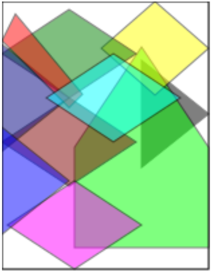
- In most cases, the dual bounding model tracks closely with the Fischetti-Luzzi during the first several seconds. It then converges to a bound and stays there. The Fischetti-Luzzi model continues to improve its dual bound.
- The models that use bisection, subdivision, and successive approximation improve that bound slightly, but not significantly.
- In the diamond packing test case, the dual bounding model represents the same problem with fewer binary variables and outperforms the Fischetti-Luzzi model.
- Without improvements in performance, the only general advantage of the dual bounding model is that it provides a set of polygon offsets corresponding to its bound.

Future Work

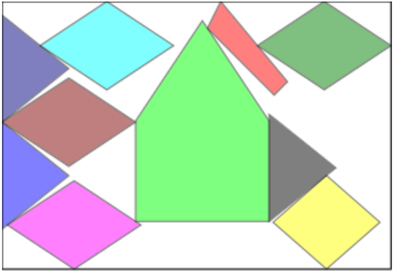
Investigate incorporating a separation algorithm based on that of Gomes and Oliveira (2006) or using one based on the NFP.

The example below uses a simple NFP-based approach. There are obviously better ways to perform separation.

Dual Bounding Output



After Separation



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